Module-1: First-Order Partial Differential Equations

1. Introduction

Many problems in mathematical, physical, engineering, biological sciences etc. deal with the formulation and the solutions of the first-order partial differential equations. These equations provide a conceptual basis which can be utilized for higher-order equation as well.

2. Classification of First-Order Equations

The most general first-order partial differential equation in two independent variables x and y is of the form

$$F(x,y,z,z_x,z_y) = 0, \quad x,y \in D \subset \mathbf{R^n} \tag{1}$$

where $z = z(x,y)$ is an unknown function of the independent variables x and y, F is a given function of arguments and $z_x = \frac{\partial z}{\partial x}$, $z_y = \frac{\partial z}{\partial y}$. Equation (1) in terms of standard notation $p = z_x$, $q = z_y$ takes the form

$$F(x,y,z,p,q) = 0, \tag{2}$$

Similarly, the most general first-order partial differential equation in three independent variables x, y, u is of the form

$$F(x,y,u,z,z_x,z_y,z_u) = 0. \tag{3}$$

In a similar way, we can construct first-order partial differential equations in any number of independent variables. However, we limit ourselves to two independent variables x, y and the dependent variable z.

First-order partial differential equations of the form (2) can be classified into four categories as follows:

(a) *Linear equation*

Equation (2) is said to be *linear* if F is linear in each of the variables z, p and q and the coefficients of these variables are functions of the independent variables x and y only. The most general first-order linear partial differential equation is of the form

$$a(x,y)p + b(x,y)q + c(x,y)z = d(x,y). \tag{4}$$

The functions a, b, c and d are assumed to be continuously differentiable. If $d(x,y) = 0$, the equation (4) is said to be *homogeneous* and if $d(x,y) \neq 0$, it is *non-homogeneous* partial differential equation.

Examples of linear first-order equations are

$$np + (x+y)q - z = e^x,$$
$$(y-z)p + (z-x)q + (x-y)z = 0.$$

(b) *Quasi-linear equation*

Equation (2) is called a *quasi-linear* first-order partial differential equation if it is linear in first-partial derivatives of the unknown function $z(x,y)$, i.e. if the equation is of the form

$$a(x,y,z)p + b(x,y,z)q = c(x,y,z). \tag{5}$$

Examples of quasi-linear equations are

$$\begin{aligned} x(z-2y^2)p + y(z-y^2-2x^2)q &= z(z-y^2-2x^2), \\ zp + q + nz^2 &= 0. \end{aligned}$$

(c) *Semi-linear equation*

The equation (2) is *semi-linear* if the coefficients of p and q are independent of z, i.e. the equation is of the type

$$a(x,y)p + b(x,y)q = c(x,y,z). \tag{6}$$

Examples of semi-linear equations are

$$\begin{aligned} (x+1)^2p + (y-1)^2q &= (x+y)z^2, \\ xp + yq &= z^2 + x^2. \end{aligned}$$

(d) *Nonlinear equation*

An equation which does not belong to above types is known as *nonlinear* first-order partial differential equation.

Examples of nonlinear equations are

$$p^3 + q^3 = 3pq,$$
$$q + xp = p^2.$$

3. Construction of First-Order Equations

First-order partial differential equations can be originated in many ways, e.g.by elimination of arbitrary constants or functions and in studying physical or social phenomena. Let us demonstrate how these equations occur.

(a) *Elimination of arbitrary constants.*

Consider a system of geometrical surfaces described by the relation

$$f(x, y, z, a, b) = 0 \tag{7}$$

involving two independent variables x and y, one dependent variable $z(x, y)$ and two arbitrary constants a, b. Differentiation of (7) with respect to x and y, we get respectively

$$f_x + pf_z = 0 \text{ and } f_y + qf_z = 0, \left(p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}\right)$$

which involve two arbitrary parameters a and b and these can be eliminated from the above set and equation (7) to obtain a first-order partial differential equation of the form

$$F(x, y, z, p, q) = 0.$$

Example 1: Formulate the partial differential equation by eliminating the arbitrary constants a and b from the equation $(x-a)^2 + (y-b)^2 + z^2 = 1$.

Solution: Differentiating the given equation $(x-a)^2 + (y-b)^2 + z^2 = 1$ with respect to x and y, we get respectively

$$x - a + zp = 0 \text{ and } y - b + zq = 0 \text{ so that } a = x + zp \text{ and } b = y + zq.$$

Substituting these values of a and b in the given equation, the required partial differential equation is $z^2(p^2 + q^2 + 1) = 1$.

(b) *Elimination of arbitrary functions.*

Let $u = u(x, y, z)$ and $v = v(x, y, z)$, where $z = z(x, y)$, be two given functions of $x,\ y,\ z$ connected by the relation

$$\phi(u, v) = 0.$$

Differentiating this relation partially with respect to x and y we get respectively

$$\begin{aligned} \frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) &= 0, \\ \text{and} \quad \frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) &= 0. \end{aligned}$$

Elimination of $\frac{\partial\phi}{\partial u}$ and $\frac{\partial\phi}{\partial v}$ between these two equations leads to

$$\frac{\partial(u, v)}{\partial(y, z)}p + \frac{\partial(u, v)}{\partial(z, x)}q = \frac{\partial(u, v)}{\partial(x, y)}$$

$$\text{i.e.} \quad Pp + Qq = R \tag{8}$$

$$\text{where} \quad P = \frac{\partial(u, v)}{\partial(y, z)}, \quad Q = \frac{\partial(u, v)}{\partial(z, x)}, \quad R = \frac{\partial(u, v)}{\partial(x, y)}. \tag{9}$$

The first-order linear equation (8) is known as *Lagrange's equation of first-order*.

Note: If the given relation between $x,\ y$ and z contains two arbitrary functions, then (excepting some cases), the partial differential equations of higher order are formed.

Example 2: Find the partial differential equation by eliminating the arbitrary function from the equation $z = f\left(\frac{xy}{z}\right)$.

Solution: Differentiating the equation $z = f\left(\frac{xy}{z}\right)$ with respect to x and y, we get respectively

$$p = f'\left(\frac{xy}{z}\right)\left(\frac{y}{z} - \frac{xy}{z^2}p\right) \quad \text{and} \quad q = f'\left(\frac{xy}{z}\right)\left(\frac{x}{z} - \frac{xy}{z^2}q\right)$$

$$\text{so that} \quad \frac{p}{q} = \frac{yz - xyp}{zx - xyq} \quad \text{i.e.} \quad px - qy = 0$$

This is the required equation.

We have already seen that a relation of the form $f(x, y, z, a, b) = 0$ leads to a partial differential equations of first-order. Such a relation containing two arbitrary constants a and b is a *solution* of the first-order partial differential equation and is called a *complete solution* or a *complete integral* of that equation.

On the other hand, any relation of the type $\phi(u,v) = 0$ involving an arbitrary function ϕ connecting two known functions $u(x,y,z)$ and $v(x,y,z)$ and providing a solution of the first-order partial differential equation is called a *general solution* or *general integral* of the equation. The general solution can also be obtained as the locus of a parametric family of curves, called *characteristics* of the envelope of the family $f(x,y,z,a,\psi(a)) = 0$, where b is supposed to be a function of a. The general solution of a first-order partial differential equation represents a family of surfaces, called *integral surfaces*.

The *singular solution* or *singular integral* is obtained from the complete integral by the elimination of arbitrary constants. Thus, if $f(x,y,z,a,b) = 0$ is the complete integral of the partial differential equation $F(x,y,z,p,q) = 0$, then the a and b - eliminant from the equations $f = 0$, $\frac{\partial f}{\partial a} = 0$ and $\frac{\partial f}{\partial b} = 0$ is the singular solution. The singular solution can also be obtained from the partial differential equation itself by elimination of p and q from the equations $F = 0$, $\frac{\partial F}{\partial p} = 0$, $\frac{\partial F}{\partial q} = 0$,

Example 3: The equation $z^2(p^2 + q^2 + 1) = c^2$, where c is constant, has a complete integral of the form $(x-a)^2 + (y-b)^2 + z^2 = c^2$, a and b being arbitrary constants. Find the singular integral and a general integral assuming $b = a$.

Solution: Differentiating the relation $f(x,y,z,a,b) = (x-a)^2+(y-b)^2+z^2-c^2 = 0$ partially with respect to a and b, we get respectively $x-a = 0$ and $y-b = 0$ i.e. $a = x$, $b = y$. Eliminating of a and b from $f(x,y,z,a,b) = 0$ gives the singular solution as $z = \pm c$.

Alternatively, We differentiate the given partial differential equation $F(x,y,z,p,q) = z^2(p^2+q^2+1)-c^2 = 0$ partially with respect to p and $q$and get respectively $pz = 0$ and $qz = 0$, i.e. $p = 0$ and $q = 0$. Eliminating p and q from $F(x,y,z,p,q) = 0$ the desired singular solution is obtained as $z = \pm c$.

Now, making $b = a$ in $f(x,y,z,a,b) = 0$ we get

$$f(x,y,z,a,a) = (x-a)^2 + (y-a)^2 + z^2 - c^2 = 0$$

Differentiating with respect to a, gives $x-a+y-a = 0$ i.e. $a = \frac{1}{2}(x+y)$. Eliminating a from $f(x,y,z,a,a) = 0$ we have $\frac{1}{2}(x-y)^2+z^2-c^2 = 0$, i.e. $(x-y)^2+2z^2 = 2c^2$ which is the required general integral,

Finally, it is important to note that the solutions of a partial differential equation are to be represented by *smooth functions*, i.e. the functions whose all derivatives exist and are continuous. However, in general, solutions are not always smooth. A solution which is not everywhere differentiable is called a *weak solution*.

4. Existence of Solutions of First-Order Partial differential Equations-Cauchy Problem

The existence of solutions of a first-order partial differential equation is not guaranteed. However, if the equation satisfies certain conditions, then the solution does exist.

Cauchy problem:

Suppose

(a) the function $x_0(\mu)$, $y_0(\mu)$ and $z_0(\mu)$ along with their first derivatives are continuous in the interval $I : \mu_1 < \mu < \mu_2$ and

(b) the function $F(x,y,z,p,q)$ is continuous in x,y,z,p,q in a region Ω of the $xyzpq$-space.

Then the problem is to establish a function $\phi(x,y)$ with the following properties :

(i) The function $\phi(x,y)$ and its derivatives $\phi_x(x,y)$ with respect to x and $\phi_y(x,y)$ with respect to y are continuous in a region R of the xy-space.

(ii) The point $\left\{x,y,\phi(x,y),\phi_x(x,y),\phi_y(x,y)\right\} \in \Omega$ and $F\left(x,y,\phi(x,y),\phi_x(x,y),\phi_y(x,y)\right) = 0, \ \forall\ x,y \in R.$

(iii) The point $\{x_0(\mu),y_0(\mu)\} \in R$ and $\phi\{x_0(\mu),y_0(\mu)\} = z_0, \ \forall\ \mu \in I.$

Geometrically, Cauchy problem can be stated as follows:

To prove the existence of a surface $z = f(x,y)$ passing through a curve Γ with parametric equation

$$x = x_0(\mu), \quad y = y_0(\mu), \quad z = z_0(\mu) \tag{10}$$

and at every point of which the direction $(p,q-1)$ of the normal is such that

$$F(x,y,z,p,q) = 0 \tag{11}$$

For the existence of a solution of the equation (11) passing through a curve Γ having equations (10), we have to make some other assumptions regarding the function F and the curve Γ. We state a theorem, known as *Cauchy-Kowalewski theorem* without proof for the existence of solution of (11).

Cauchy-Kowalewski theorem

Suppose a function $g(y)$ and all its derivatives are continuous for $|y-y_0|<\delta$, x_0 is a given number and $z_0 = g(y_0)$, $q_0 = g'(y_0)$. Also, the $f(x,y,z,q)$ and all its partial derivatives are continuous in a region S defined by

$$|x-x_0|<\delta, \quad |y-y_0|<\delta, \quad |z-z_0|<\delta.$$

Then there exists a unique function $\phi(x,y)$ such that

(i) $\phi(x,y)$ and all its partial derivatives are continuous in a region R defined by

$$|x-x_0|<\delta_1, \quad |y-y_0|<\delta_2;$$

(ii) for all $(x,y)\in R$, $z=\phi(x,y)$ is a solution of the equation $p=f(x,y,z,q)$;

(iii) for all values of y in the interval $|y-y_0|<\delta$, $\phi(x_0,y)=g(y)$.

5. Compatible Systems of First-Order Equations

Two first-order partial differential equations

$$F(x,y,z,p,q)=0 \text{ and } G(x,y,z,p,q)=0 \tag{12}$$

are said to be *compatible* if a solution of the former satisfies the latter and conversely.

Let us assume that $\mathrm{J} = \frac{\partial(F,G)}{\partial(p,q)} \neq 0$ so that the equations (12) can be solved for p and q in the form

$$p=p(x,y,z), \qquad q(q(x,y,z).$$

Now the condition of compatibility for the equations (12) is that $dz = pdx + qdy$ is integrable and it is possible provided $\mathbf{X}.(\nabla\times\mathbf{X})=0$, where $\mathbf{X}=(p,q-1)$, i.e. if

$$p\frac{\partial q}{\partial z} - q\frac{\partial p}{\partial z} + \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} = 0,$$

$$\text{or,} \qquad q_x + pq_z = p_y + qp_z. \tag{13}$$

Now, differentiating the first equation of (12) partially with respect to x and z, we get respectively

$$F_x + F_p p_x + F_q q_x = 0$$
$$\text{and} \qquad F_z + F_p p_z + F_q q_z = 0.$$

Multiplying the second equation of these by p and adding the result with the first equation, we obtain

$$F_x + pF_z + F_p(p_x + pp_z) + F_q(q_x + pq_z) = 0. \tag{14a}$$

Similarly, from the second equation of (12) we get

$$G_x + pG_z + G_p(p_x + pp_z) + G_q(q_x + pq_z) = 0. \tag{14b}$$

Elimination of $p_x + pp_z$ between the equations (14) leads to

$$\frac{\partial(F,G)}{\partial(x,p)} + p\frac{\partial(F,G)}{\partial(z,p)} - \frac{\partial(F,G)}{\partial(p,q)}(q_x + pq_z) = 0$$

so that $$q_x + pq_z = \frac{1}{J}\left[\frac{\partial(F,G)}{\partial(x,p)} + p\frac{\partial(F,G)}{\partial(z,p)}\right]. \tag{15a}$$

Similarly, differentiating the equations (12) with respect to y and z and then proceeding as above, we obtain

$$p_y + qp_z = -\frac{1}{J}\left[\frac{\partial(F,G)}{\partial(y,q)} + q\frac{\partial(F,G)}{\partial(z,q)}\right]. \tag{15b}$$

Using (15), it follows from (13)

$$\frac{\partial(F,G)}{\partial(x,p)} + \frac{\partial(F,G)}{\partial(y,q)} + p\frac{\partial(F,G)}{\partial(z,p)} + q\frac{\partial(F,G)}{\partial(z,q)} = 0$$

which, in short, is written as $[F,G] = 0.$ (16)

Example 4: Show that the equations $xp = yq$ and $z(xp + yq) = 2xy$ are compatible.

Solution: Let $F(x,y,z,p,q) = xp - yq = 0$ and $G(x,y,z,p,q) = z(xp+yq) - 2xy = 0$

$$
\begin{aligned}
[F,G] &= \frac{\partial(F,G)}{\partial(x,p)} + \frac{\partial(F,G)}{\partial(y,q)} + p\frac{\partial(F,G)}{\partial(z,p)} + q\frac{\partial(F,G)}{\partial(z,q)} \\
&= \left(\frac{\partial F}{\partial x}\frac{\partial G}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial G}{\partial x}\right) + \left(\frac{\partial F}{\partial y}\frac{\partial G}{\partial q} - \frac{\partial F}{\partial q}\frac{\partial G}{\partial y}\right) \\
&\quad + p\left(\frac{\partial F}{\partial z}\frac{\partial G}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial G}{\partial z}\right) + q\left(\frac{\partial F}{\partial z}\frac{\partial G}{\partial q} - \frac{\partial F}{\partial q}\frac{\partial G}{\partial z}\right) \\
&= \Big\{p(zx) - x(zp-2y)\Big\} + \Big\{(-q)zy - (-y)(zq-2x)\Big\} \\
&\quad + p\Big\{0.(zx) - x(zp+yq)\Big\} + q\Big\{0.(zq) - (-y)(xp+yq)\Big\} \\
&= 2xy - 2xy + (xp+yq)(-px+qy) \\
&= 0 \quad (\because px = qy).
\end{aligned}
$$

Thus the given equations are compatible.